Exercise 17

Prove the identity.

$$\tanh(\ln x) = \frac{x^2 - 1}{x^2 + 1}$$

Solution

Use the definitions listed on page 259.

$$\begin{aligned}
\tanh(\ln x) &= \frac{\sinh(\ln x)}{\cosh(\ln x)} \\
&= \frac{\frac{e^{\ln x} - e^{-\ln x}}{2}}{\frac{e^{\ln x} + e^{-\ln x}}{2}} \\
&= \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}} \\
&= \frac{e^{\ln x} - e^{\ln x^{-1}}}{e^{\ln x} + e^{\ln x^{-1}}} \\
&= \frac{x - x^{-1}}{x + x^{-1}} \\
&= \frac{x - \frac{1}{x}}{x + \frac{1}{x}} \times \frac{x}{x} \\
&= \frac{x^2 - 1}{x^2 + 1}
\end{aligned}$$